ON NECESSARY CONDITIONS FOR A STEP CHANGE IN THE COMBUSTION VELOCITY OF CONDENSED SYSTEMS

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UDC 536.46

The ordinary formulation of the problem for the theory of combustion of condensed systems is that for a given external effect on the system (the pressure p, the light flux q, the velocity of erosive gas flux g, etc.), it is required to determine the combustion velocity u. The solution of such a problem (we call it direct) can be treated as finding sufficient conditions to realize a given change in the combustion velocity. Besides the direct problem, the inverse problem is of interest: to find the necessary conditions to realize a given law of time variation of the combustion velocity u(t).

Under the assumption of the Ya. B. Zel'dovich [1] combustion model, an exact solution of the inverse problem is obtained herein for the case when the change in combustion velocity u with time t is given as a step change in the velocity from the stationary value u_0 for $t<t_0$ to the stationary value u_1 for $t>t_0$.

It has been established that the form of the law of time variation in the pressure p(t) specifying a step change in the combustion velocity u(t) depends essentially on the difference between the velocity ratio $z=u_1/u_0$ and the Ya. B. Zel'dovich criterion [2]

$$k=(T_1-T_0)\left(\frac{\partial \ln u}{\partial T_0}\right)_p \qquad (0<k<1)$$

where T_0, T_1 are the initial temperature and the surface temperature of the system, respectively. Thus, if $z>k$, then the combustion mode under consideration is stable at any instant and can exist infinitely long. If $0<z\leq k$, then at some time $t=t_1$ ($t_1>t_0$), combustion becomes unstable, where the loss in stability sets in more rapidly, the less the quantity z differs from k, and the later, the greater the quantity $k-z$.

1. FORMULATION OF THE PROBLEM

As Ya. B. Zel'dovich and B. V. Novozhilov [1-3] have shown, nonstationary combustion processes of condensed systems can be computed sufficiently accurately by solving the equation of heat conduction in the condensed phase (k-phase)

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=\varkappa\frac{\partial^2 T}{\partial x^2} \qquad (-\infty<x\leqslant 0) \tag{1.1}$$

with the initial and boundary conditions

$$T(x, t=0)=T_0(x), \quad T(-\infty, t)=T_0(-\infty), \quad T(0, t)=T_1 \tag{1.2}$$

and the condition that the combustion velocity u and the surface temperature T_1 depend on the temperature gradient at the surface

$$f=\left.\frac{\partial T}{\partial x}\right|_{x=0}$$

and the external factors according to the known laws

$$u=u(f;\ p, g, \ldots), \qquad T_1=T_1(f; p, g, \ldots) \tag{1.3}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 57-64, November-December, 1971. Original article submitted February 1, 1971.

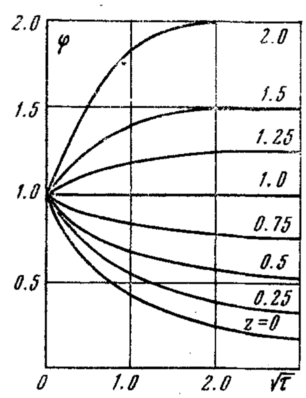

Fig. 1

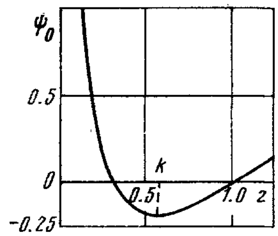

Fig. 2

These laws are found from converting the stationary dependences of the combustion velocity and surface temperature on the initial temperature of the k-phase $T_0(-\infty)$, and parameters of the type of the pressure p, the erosive velocity g, etc. In solving direct problems of combustion theory, the system (1.1)-(1.3) is closed by assigning time dependences of the pressure, the erosive flux, etc.:

$$p = p(t), \quad g = g(t), \ldots \tag{1.4}$$

In the case of the inverse problem, it is necessary to give the time dependence of the combustion velocity

$$u = u(t) \tag{1.5}$$

The solution of the heat conduction equation is needed to find just the functional relationship among the surface temperature T_1, the temperature gradient at the surface f, and the combustion velocity u at an arbitrary time t in both the direct and inverse problems of the theory of nonstationary combustion:

$$F\left(T(0, t); \left.\frac{\partial T}{\partial x}\right|_{x=0}; u(t); t\right) = 0 \tag{1.6}$$

At the same time, the solution of the heat conduction equation T(x, t) itself carries so great an amount of information that is useless in this case. However, this disadvantage can easily be overcome by applying a Fourier transform to (1.1) under the conditions (1.2), as has been shown in [4]. The relationship (1.6) obtained as a result is

$$T_1 - T_0(-\infty) - \frac{1}{\sqrt{\pi \varkappa t}} \int\limits_{-\infty}^{0} [T_0(x) - T_0(-\infty)] \exp\left[-(4\varkappa t)^{-1} \left(\int\limits_0^t u dt + x\right)^2\right] dx$$
$$- \int\limits_0^t \exp\left[-\frac{U^2}{4\varkappa(t-t')}\right] \frac{T_1 - T_0(-\infty)}{\sqrt{\pi\varkappa(t-t')}} \left[\frac{\varkappa f(t')}{T_1 - T_0(-\infty)} - u(t') + \frac{U}{2(t-t')}\right] dt' = 0 \tag{1.7}$$

where

$$U = \int\limits_{t'}^{t} u dt \tag{1.8}$$

In the case of the Ya. B. Zel'dovich combustion model, when $T_1 = \text{const}$ the integral relationship (1.7) can be simplified by considering that

$$\int\limits_0^t \exp\left[-\frac{U^2}{4\varkappa(t-t')}\right] \frac{1}{\sqrt{\pi\varkappa(t-t')}} \left[\frac{U}{2(t-t')} - u\right] dt' = -\operatorname{erf}\left[2^{-1}(\varkappa t)^{-1/2} \int\limits_0^t u dt\right],$$
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-\alpha^2} d\alpha \tag{1.9}$$

Then

$$1 + \operatorname{erf}\left[2^{-1}(\varkappa t)^{-1/2}\int\limits_0^t u dt\right] - \frac{1}{\sqrt{\pi\varkappa t}} \int\limits_{-\infty}^{0} \frac{T_0(x) - T_0(-\infty)}{T_1 - T_0(-\infty)} \exp\left[-(4\varkappa t)^{-1} \times \left(\int\limits_0^t u dt + x\right)^2\right] dx$$

$$= \int_0^t \exp\left[-\frac{U^2}{4\varkappa(t-t')}\right] \frac{\varkappa f(t')}{(T_1 - T_0(-\infty))\sqrt{\pi\varkappa(t-t')}}\, dt' \quad (1.10)$$

In this form, the relationship (1.7) is particularly convenient for the solution of inverse problems since in this case u(t) and respectively, U(t, t') in (1.8) are known functions of time, and the integral equation (1.10) in f is a Volterra equation of the first kind. Substituting the function $f(t)$ found from this equation into the first equation of (1.3), a relationship can be obtained between the pressure p, the flux velocity g, and the other external parameters

$$G(t, p, g, \ldots) = 0 \quad (1.11)$$

This relationship is a necessary condition for the realization of a given change in the velocity u(t).

If all the external factors, with the exception of one, say p, are fixed, (1.11) will yield the pressure variation law p(t) needed for a given change in the combustion velocity u(t).

2. TRANSIENT MODE

Let us see the pressure variation law for which the combustion velocity at some time varies by a jump from the stationary value $u=u_0$ to the stationary value $u=u_1$. Let the time at which the velocity jump occurs be zero.

Then u(t) becomes

$$u(t) = \begin{cases} u_0 & \text{at} \quad t \leqslant 0 \\ u_1 & \text{at} \quad t > 0 \end{cases} \quad (2.1)$$

Hence, the corresponding temperature distribution at the time t=0 is

$$T_0(x) = T_0 + (T_1 - T_0)\exp\left(\frac{u_0}{\varkappa}x\right) \quad (2.2)$$

Let us introduce the dimensionless variables

$$\tau = \frac{u_0^2}{\varkappa}t, \quad \xi = \frac{u_0}{\varkappa}x, \quad \theta = \frac{T - T_0}{T_1 - T_0} \quad (2.3)$$

$$\varphi = \frac{\varkappa f}{u_0(T_1 - T_0)}, \quad v = \frac{u}{u_0}, \quad \eta = \frac{p}{p_0}$$

where p_0 is the stationary pressure corresponding to the velocity u_0.

Substituting (2.2) and (2.1) into (1.10) and transforming to the new variables (2.3), we obtain

$$1 + \operatorname{erf}\left(\tfrac{1}{2}z\sqrt{\tau}\right) - e^{-(z-1)\tau}\left\{1 - \operatorname{erf}\left[\left(1 - \tfrac{1}{2}z\right)\sqrt{\tau}\right]\right\} = \int_0^\tau \exp\left[-\frac{z^2}{4}(\tau - \tau')\right]\frac{\varphi(\tau')}{\sqrt{\pi(\tau - \tau')}}\,d\tau' \quad (2.4)$$

where

$$z = u_1/u_0 \quad (2.5)$$

Equation (2.4) is a Volterra integral equation of the first kind with kernel dependent on the difference between the arguments $\tau - \tau'$. The technique of solving such equations by using the Laplace transform has been developed well [5].

Let us introduce the notation

$$G(\tau) = 1 + \operatorname{erf}\left(\tfrac{1}{2}z\sqrt{\tau}\right) - e^{-(z-1)\tau}\left\{1 - \operatorname{erf}\left[\left(1 - \tfrac{1}{2}z\right)\sqrt{\tau}\right]\right\} \quad (2.6)$$

$$K(\tau) = \frac{1}{\sqrt{\pi\tau}}e^{-\frac{1}{4}z^2\tau} \quad (2.7)$$

In this notation, (2.4) becomes

$$G(\tau) = \int_0^\tau K(\tau - \tau')\,\varphi(\tau')\,d\tau' \quad (2.8)$$

Let

$$\int_0^\tau \varphi(\tau')\,d\tau' = \Phi(\tau) \tag{2.9}$$

Then, applying the Laplace transformation to (2.8), we obtain an algebraic relationship for the transforms

$$f(s) = \frac{1}{sk(s)}\,g(s) \tag{2.10}$$

where

$$f(s) = \int_0^\infty e^{-s\tau}\Phi(\tau)\,d\tau \tag{2.11}$$

$$g(s) = \int_0^\infty e^{-s\tau}G(\tau)\,d\tau \tag{2.12}$$

$$k(s) = \int_0^\infty e^{-s\tau}K(\tau)\,d\tau \tag{2.13}$$

Evaluating k(s) by using (2.7), let us represent (2.10) as

$$f(s) = \left(\frac{1}{s}\,\frac{{}^1\!/_4 z^2}{\sqrt{s+{}^1\!/_4 z^2}} + \frac{1}{\sqrt{s+{}^1\!/_4 z^2}}\right)g(s) \tag{2.14}$$

Now, applying the inverse Laplace transformation to (2.14), we obtain the expression for the original $\Phi(\tau)$

$$\Phi(\tau) = \int_0^\tau \left[\frac{z}{2}\operatorname{erf}\left(\frac{z}{2}\sqrt{\tau'}\right) + \frac{1}{\sqrt{\pi\tau'}}e^{-{}^1\!/_4 z^2\tau'}\right]G(\tau-\tau')\,d\tau' \tag{2.15}$$

Differentiating (2.15) with respect to the time τ, and taking into account that $G(0)=0$ according to (2.6), we obtain the expression for the gradient φ

$$\varphi(\tau) = \int_0^\tau \left[\frac{z}{2}\operatorname{erf}\left(\frac{z}{2}\sqrt{\tau'}\right) + \frac{1}{\sqrt{\pi\tau'}}e^{-{}^1\!/_4 z^2\tau'}\right]G'(\tau-\tau')\,d\tau' \tag{2.16}$$

Evaluating $G'(\tau-\tau')$ from (2.6) and substituting into (2.16), we obtain a sufficiently simple expression for the temperature gradient by using simple manipulations

$$\varphi(\tau) = {}^1\!/_2 z\,[1+\operatorname{erf}({}^1\!/_2 z\sqrt{\tau})] + (1-{}^1\!/_2 z)\,e^{-(z-1)\tau}\{1-\operatorname{erf}[(1-{}^1\!/_2 z)\sqrt{\tau}]\}, \quad \tau \geqslant 0 \tag{2.17}$$

Let us note that G. M. Makhviladze [6] obtained (2.17) independently and by another method.

The family of curves $\varphi(\tau)$ is pictured in Fig. 1 for different values of the parameter z. All the curves possess the following property:

$$\begin{aligned} &\varphi(\tau) \to 1 \quad \text{at} \quad \tau \to 0 \ \text{and any} \quad z \\ &\varphi(\tau) \to z \quad \text{at} \quad \tau \to +\infty \end{aligned} \tag{2.18}$$

Therefore, as the combustion velocity changes by a jump from v = 1 to v = z, the temperature gradient φ at the k-phase surface changes continuously and monotonically and in time emerges into the stationary mode $\varphi = z$ corresponding to the stationary level of the combustion velocity v = z.

3. PRESSURE VARIATION LAW

In order to obtain the appropriate pressure variation law, it is necessary to assign some kind of relationship between the combustion velocity v, the pressure η, and the gradient φ in conformity with (1.3). Assuming an exponential dependence of the combustion velocity on the initial temperature, and a power-law dependence on the pressure $u \sim p^\nu e^{\alpha T_0}$, we obtain

$$v = \exp\,(k\,(1-\varphi/v))\,\eta^\nu \tag{3.1}$$

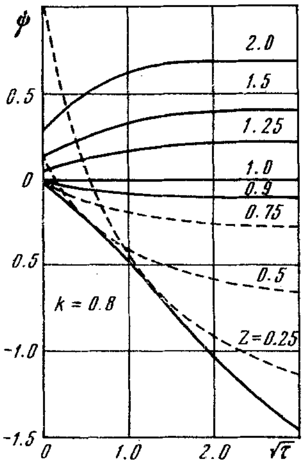

Fig. 3

where

$$k = \alpha (T_1 - T_0) \qquad (0 < k < 1)$$

Substituting the condition v=z and the expression for φ into (3.1), we obtain from (2.5) for the pressure

$$\psi \equiv \ln (\eta^{\nu}) = kz^{-1}\varphi(z, \tau) + \ln z - k \tag{3.2}$$

Let us investigate the nature of the behavior of the curves (3.2). For any z the curves $\psi(z, \tau)$ are monotone functions of the time since the gradients $\varphi(z, \tau)$ are monotone, and in time they emerge into the stationary mode $\psi = \ln z$. As the time τ tends to zero on the right, the curves $\psi(z, \tau)$ tend, according to (2.7), to the limit values

$$\psi_0(z) \equiv \psi(z, +0) = (kz^{-1} - k + \ln z) \tag{3.3}$$

There is a minimum among all the values of $\psi_0(z)$ which is reached at z=k:

$$\min_z \psi_0(z) = \psi_0(k) = 1 - k + \ln k < 0 \tag{3.4}$$

A graph of the initial pressure jump as a function of the velocity jump $\psi_0(z)$ is shown in Fig. 2. Two pressure change curves $\psi_1(z_1, \tau)$ and $\psi_2(z_2, \tau)$ correspond to each initial jump $\psi_0(z)$ ($\infty > \psi_0 \geq 1 - k + \ln k$), with the exception of $\psi_0(k)$ where one corresponds to $z_1 < k$ and the other to $z_2 > k$. In this connection, the family of pressure curves $\psi(z, \tau)$ can be separated into two subfamilies (Fig. 3). The subfamily $\psi_2(z, \tau)$ ($z > k$) of the curves is characterized by the fact that a larger value of the pressure ψ corresponds to a larger value of the velocity v at the initial instant $\tau = +0$ (exactly as later). The subfamily ψ_1 consists of curves having a value ψ_0 at the time $\tau = +0$, which will be greater, the smaller the quantity z. The pressure curve $\psi(k, \tau)$ should be referred to second family for reasons which will be clarified later.

The subfamily of curves $\psi_1(z, \tau)$ has an envelope determined from the system of equations

$$\begin{aligned} \psi &= kz^{-1}\varphi(z, \tau) + \ln z - k \\ \psi_z' &= kz^{-1}\varphi_z'(z, \tau) - kz^{-2}\varphi(z, \tau) + z^{-1} = 0 \end{aligned} \tag{3.5}$$

For this it is necessary to express z as a function of τ from the second equation of the system (3.5), and to substitute the expression obtained for $z(\tau)$ into the first equation

$$\psi^* = \frac{k}{z(\tau)}\varphi(z(\tau), \tau) + \ln z(\tau) - k \tag{3.6}$$

However, it is difficult to obtain an explicit expression for the envelope ψ^* at any instant since the second equation of the system (3.5) in z is transcendental:

$$\frac{k}{z^2}\left\{\frac{z}{k} + z\sqrt{\frac{\tau}{\pi}}e^{-1/_4 z^2\tau} - e^{(1-z)\tau}\left[1 - \operatorname{erf}\left(\left(1 - \frac{z}{2}\right)\sqrt{\tau}\right)\right]\left(\tau\frac{z^2}{2} - \tau z + 1\right)\right\} = 0 \tag{3.7}$$

Hence, let us limit ourselves to the investigation of the behavior of the envelope for sufficiently small and large times. For small times $\tau \ll 1$ the asymptotic expression for the pressure ψ is

$$\psi = kz^{-1}[1 - 2(1 - z)\sqrt{\pi^{-1}\tau} + (1 - z)(1 - {}^1/_2 z)\tau] + \ln z - k \tag{3.8}$$

We hence find $z(\tau)$ from the second equation of (3.5)

$$z(\tau) = k[1 - 2\sqrt{\pi^{-1}\tau} + (1 - {}^1/_2 k^2)\tau] \tag{3.9}$$

and the asymptotic expression for the envelope from the first

$$\psi^* = (1 - k + \ln k) - (1 - k)2\sqrt{\pi^{-1}\tau} + (1 - 2\pi^{-1} + {}^1/_2 k^2 - {}^3/_2 k)\tau \tag{3.10}$$

As is seen from (3.8) and (3.9), the envelope ψ^* at the time $\tau = +0$ is tangent to the pressure curve ψ corresponding to z=k, and later has a tangent to the curves corresponding to z<k (i.e., to the curves of the first subfamily $\psi_1(z, \tau)$). Hence, the tangent of the curves $\psi_1(z, \tau)$ to the envelope will start later, the smaller the quantity z to which they correspond. The same picture is indeed conserved for large times $\tau \gg 1$. An asymptotic expression for ψ_1 is valid for large times:

$$\psi = \ln z + \frac{k}{z}\frac{2}{\sqrt{\pi}}\frac{1}{\tau^{3/2}}e^{-1/_4 z^2\tau}\frac{1 - z}{z^2(1 - {}^1/_2 z)^2}, \qquad z \neq 0 \tag{3.11}$$

We obtain z(τ) from the second equation of (3.5):

$$z(\tau) = 2C / \sqrt{\tau} \tag{3.12}$$

where the constant C is determined from the equation

$$Ce^{C^2} = k / 2\sqrt{\pi} \tag{3.13}$$

Substituting (3.12) into the first equation of (3.5), we obtain an expression for the envelope at large times

$$\psi^* = \ln\left(\frac{2C}{\sqrt{\tau}}\right) + \frac{1}{2C^2} \tag{3.14}$$

As follows from (3.12), (3.14), the envelope as well as the pressure curves themselves near the envelope vary similarly to the known self-similar modes [2, 7] at large times, for which

$$\left(\frac{p}{p_0}\right)^\nu \sim \frac{D}{\sqrt{t}}$$

The combustion velocity is hence constant, while $u(t) \sim 1/\sqrt{t}$ in the mentioned self-similar modes.

4. STABILITY OF THE NONSTATIONARY MODES

As has been mentioned above, a solution of the inverse problem of nonstationary combustion theory (in a particular case) is the pressure variation law $\psi(\tau)$ which assures a change in the combustion velocity according to a given law $v(\tau)$ for given initial conditions $\theta_0(\xi)$. Therefore

$$\psi(\tau) = \psi(v(\tau), \theta_0(\xi), \tau) \tag{4.1}$$

Let us assume that the mentioned quantities acquire small increments $\delta\psi(\tau)$, $\delta v(\tau)$, $\delta\theta_0(\xi)$, then by variating the expression (4.1), we obtain a relationship between the increments

$$\delta\psi(\tau) = \psi_v'(\delta v) + \psi_{\theta_0}'(\delta\theta_0) \tag{4.2}$$

where ψ_v' and ψ'_{θ_0} are the first variations of the functional ψ. The relationship (4.2) permits answering the question of whether a given process is stable to perturbations of any kind of the variables ψ, v or $\theta_0(\xi)$.

Let us examine the stability of the mode investigated above. Let us assume that the initial temperature distribution experiences no perturbations, but the pressure acquires the small perturbation $\delta\psi(z, \tau)$. Then the combustion velocity receives the perturbation $\delta z(\tau)$, which depends on $\delta\psi$ as follows in conformity with (3.2) and (4.2)

$$\delta z(\tau) = \left[\frac{k}{z}\varphi_z'(z, \tau) - \frac{k}{z^2}\varphi(z, \tau) + \frac{1}{z}\right]^{-1}\delta\psi \tag{4.3}$$

The mode z(τ) will be Lyapunov-stable to perturbations $\delta\psi(\tau)$ if for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $\|\delta\psi\| < \delta(\varepsilon)$, then $\|\delta z(\tau)\| \leqslant \varepsilon$. As the norm of the function, let us select its maximum absolute value $\|f(\tau)\| = \max_\tau |f(\tau)|$. Then the stability condition for z(τ) will be satisfied if the factor in (4.3) is bounded in absolute value. This requirement is satisfied for the curves of the second subfamily $\psi_2(z, \tau)$, $(z > k)$ and is not satisfied for curves of the first subfamily $\psi_1(z, \tau)$ since in this case the coefficient $(kz^{-1}\varphi_z - kz^{-2}\varphi + z^{-1})^{-1}$ becomes infinite at the time the curve $\psi_1(z, \tau)$ is tangent to the envelope $\psi^*(\tau)$.

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